Use of genetic decompositions to scaffold the development of a structurally sequenced curriculum for mathematics acceleration

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The authors have collaboratively used a graphical language to describe their shared knowledge of a small domain of mathematics, which has in turn scaffolded their re-development of a related curriculum for mathematics acceleration. This collaborative use of the graphical language is reported as a simple descriptive case study. This leads to an evaluation of the graphical language's usefulness as a tool to support the articulation of the structure of mathematics knowledge. In turn, implications are drawn for how the graphical language may be utilised as the detail of the curriculum is further elaborated and communicated to teachers.

Keywords: acceleration, curriculum design, genetic decomposition, mathematics, structured sequence

Introduction

The 'Accelerating the mathematics learning of low socio-economic status junior secondary students' project, more simply referred to as 'XLR8', aims to develop theory and practice to increase the life-chances of mathematically under-performing junior-secondary students (Cooper, Nutchey, & Grant, 2013). XLR8 is based upon a lineage of cognitivist ideas (Warren, 2008; Warren & Cooper, 2009), adopting a structural, cognitivist perspective, framed by such works as Sfard (1991), English and Halford (1995), and Hiebert and Carpenter (1992). The project also extends the practices developed over an extended period of time by the authors, in particular the work of Cooper. Central to achieving its aim, the XLR8 project is developing a two year curriculum for Year 8 and Year 9 students (ages 12-14 years) that follows a structured sequence of ideas from foundational lower-primary concepts through to those more typically associated with Year 9 students, such that these students might be prepared to enter Year 10 and later schooling with an appropriate level of mathematical understanding. This new curriculum is characterised as 'vertical', based upon the notion that deep understanding and gestalt jumps (i.e., acceleration) of learning can occur when intent focus is placed upon the structure of mathematical concepts within a relatively narrow area of mathematics.

The first year of the XLR8 curriculum has been developed and trialled in the Australian 2013 school year. The trialled version of the curriculum has a very strong vertical alignment, with a focus placed upon number, operation and algebra concepts. Researcher observations, interviews of the teachers delivering the curriculum and analysis of student pre-post test data all suggest that, whilst the vertical curriculum has achieved some of its goals regarding acceleration, the narrow focus upon number and algebra-related concepts may be limiting the meaningful contextualisation and thus deeper understanding of the mathematical concepts. Also, observations of lessons in each of the XLR8 classrooms suggest that teachers have struggled to fully grasp the organisation of the structured sequence of the XLR8 curriculum, and hence have not been able to plan their teaching to fully explore that structure and achieve the expected acceleration. For these reasons, the organisation of the XLR8 curriculum has been reviewed with the view to more clearly identifying and articulating the organisation of mathematical ideas explored in the structured sequence.

To facilitate the process of review, in particular the exploration and documentation of the organisation of mathematical ideas, the graphical language developed by Nutchey (2011a, 2011b) and the associated processes to develop genetic decompositions of a domain of mathematics have been employed. This has led to the re-formulation of the two year XLR8 curriculum to take a broader focus regarding the organisation of mathematical ideas, whilst still retaining the underlying principles that are believed to lead to accelerated learning. This paper reports upon the use of Nutchey's graphical language and the impact that this has had upon the clarification of the XLR8 structured sequence.

Literature Review

Warren and Cooper (2009) described mathematical understanding to be the connectedness of a learner's internal mental models (or mathematical ideas). In turn, the development of such a connected schema is via cognitive processes that determine the structural similarities and differences between mental models which in turn lead to the construction of more abstract and organising mathematical ideas. Central to Warren and Cooper's work was the cognitive interplay between what they identified as models and representations. In their words, "models are ways of thinking about abstract concepts" and "representations are the various forms of the models" (Warren & Cooper, 2009, p. 78). Warren and Cooper augmented their theory regarding the role of representations to include: Bruner's (1966) enactive-iconic-symbolic representation sequence; Dreyfus' (1991) sequencing of representation use; Filloy and Sutherland's (1996) notions of translation (the use of increasingly abstract representations) and abstraction (the activity-based construction of higher level mathematical constructs); and Duval's (1999) notions regarding the importance of the coordinated use of representations and verbal language. While not explicitly identified, Payne and Rathmell's (1975) assertions regarding the significance of verbal language when coordinating the use of representations were also evident in Warren and Cooper's conceptual framework. Warren and Cooper's identification and description of the cognitive interplay between concepts (both more informal models and more formal mathematical principles) is consistent with Sfard's (1991) reification theory, Gray and Tall's (1994) notion of procept and, more fundamentally, Piaget's (1977/2001) theory of reflective abstraction. Piaget proposed reflective abstraction as a process of accommodation by adaptation that is sufficiently powerful to describe a learner's entire conceptual development in mathematics. Five specific processes of reflective abstraction are noted in Piaget's work (Dubinsky, 1991): interiorisation, coordination, encapsulation, generalisation and reversal.

The interaction between actors in the social learning milieu (including teacher-student interaction) scaffolds conceptual development. Adapting Popper's (1978) three-world model of knowledge as a lens through which to consider theories of learning in mathematics, Nutchey (2011b) has differentiated the 'in-the-head' understanding of the individual from the stated and shared knowledge of the community. Nutchey's graphical language allows for the development of network-like models of shared knowledge which are referred to as genetic decompositions, a term borrowed from the work of Dubinsky (1991). Based upon Piaget's five processes of reflective abstraction, a genetic decomposition allows for the concepts of a domain to be identified and organised using associations that describe these cognitive processes. The formulated model of shared knowledge within a community can then be analysed to determine an instructional sequence for exploring the structure and thus developing the learners' understanding as they become members of the community. This model formulation in preparation for its analysis to yield a structured sequence is the subject of this paper and lies at the heart of the XLR8 project.

Methodology

This paper presents a simple case-study of two researchers (the first two authors) adopting a tool (the graphical language) to help scaffold their refinement of the XLR8 curriculum. The nature of this case study can be described using Thomas' (2011) four-part case study framework that involves classifying the case study in terms of subject, purpose, approach and process. This is a local knowledge case: the subject of the case is a team of researchers who are conducting and self-reporting upon their work as they bring together their knowledge regarding curriculum design and the use of a tool to describe shared mathematical knowledge. Nutchey has claimed (2011b) that the graphical language may be useful to help educators express their understanding of mathematics and in turn design useful instruction. Thus, the purpose of this study is evaluative, seeking to understand the impact that the use of the graphical language has upon clarifying and consolidating the researchers' knowledge of the domain and the development of the structured sequence-based curriculum. This evaluative activity was secondary to the researchers' core activity, which was to re-develop the curriculum. It is then clear that the approach taken in this case study is one of description: the study seeks to document the researchers' use of the graphical language in the context of the XLR8 curriculum design. The process of developing this single-case study was retrospective: The use of the graphical language to re-develop the curriculum was conducted over a period of 2-3 weeks in late 2013, shortly after which the researchers reflected upon and discussed the process of curriculum re-development and the use of the graphical language, and subsequently constructed the evaluative account of their experience. This collaborative construction of the account is consistent with the social-constructivist paradigm that underlies the graphical language and knowledge-modelling process.

Discussion

The following paragraphs provide a chronological, retrospective account of the graphical language's use. Embedded in that account are comments regarding how the use of the graphical language shaped the researchers' understanding and expression of the organisation of mathematical knowledge. Following the account, a broader evaluation of the graphical language's use is provided.

The collaborative construction of genetic decompositions aimed to describe the organisation of mathematical ideas typically encountered in Australian lower-primary (nominally Year 4) through to junior secondary (end of Year 9) classes. In preparation for this, the collaborators reviewed a variety of curriculum materials, including the content descriptors of the Australian Curriculum: Mathematics, previously developed teaching guides for the XLR8 project and materials from predecessor projects. They also reflected upon their own ideas and opinions regarding the organisation of mathematical ideas, based upon their teaching experiences and their understanding of relevant mathematics education literature.

The graphical language was developed by the first author. The second author had some basic knowledge of the language and its constructs, but had only limited experience in its use. Thus, the first element of the collaborative effort was to re-introduce the second author to the language and its use. In particular, the introduction focussed upon the use of the inheritance association (to describe a family of similar concepts), the aggregation association (to describe a concept as the coordination of parts) and the inversion association (to identify inversely-related concepts). These three associations can be used to describe the cognitively-based connections between domain concepts. In the graphical language, a concept is defined as a principle, fact or process: no distinction is made regarding the abstractness of the mathematical idea, concepts simply identify ideas that can be used to solve problems and which can be represented in various ways.

To facilitate the collaborative construction of genetic decompositions, sheets of A1-sized paper, sticky notes and pencils were used. The general process was to firstly identify a set of concepts, writing each on a separate sticky-note. These were then attached to an A1 sheet, and arranged in a hierarchy (top-most was most abstract or complex concept). Using pencil, the associations that defined this hierarchy (i.e., inheritance and aggregations) were drawn in. As needed, inversion associations were added to the genetic decompositions to highlight opposite concepts. This process was iterative: the post-it notes could be freely moved pre-positioned and the associations erased and re-drawn. In some cases, genetic decompositions were abandoned in favour of starting from scratch: the earlier genetic decomposition having served the purpose of scaffolding a conversation regarding knowledge structure.

Attention was first given to number-related concepts, including the types of numbers (e.g., wholes, fractions, integers), the relationships between these types (e.g., simple, mixed and improper are all more specific forms of common fraction), the underlying concepts of the number system (e.g., place-value) and the meanings that can be associated with a number (e.g., cardinality, ordinality). Intertwined with this analysis of number was consideration of what can be done with number, including counting, operations (including solving for numbers with unknown value) and eventually algebra using numbers of variable value. Also, as fractional numbers were considered, measurement-related concepts were also considered. Thus, during this activity, four genetic decompositions were developed in parallel: number; operations and algebra; measurement; and geometry (since, in part, geometry provides a basis for measuring). The development of these four genetic decompositions was iterative: not only did the consideration of a genetic decomposition's structure lead to its refinement, it may have also led to the refinement of another genetic decomposition since there were concepts in common For example, the ideas that define the number-as-measure cardinal meaning of number were documented in the number genetic decomposition, but that meaning was then applied to the measurement of various attributes, some of which were geometric). At one point, the defining structure of number-as-measure appeared more completely within the measurement genetic decomposition (as our focus was upon what can be measured and how is this done), but that structure was then able to be transferred to the number genetic decomposition and led to its refinement. It should be noted that genetic decompositions for statistics and probability-related concepts were produced, but this was done after the number, algebra, measurement and geometry genetic decompositions reached a relatively stable state. Excerpts of the genetic decompositions related to number, operations and algebra are re-produced in Figure 1.

An interesting observation was made during the genetic decomposition construction process. On several occasions, the second author, who has substantial experience in primary mathematics education, discussed the concepts and their organisation with respect to one another in terms of the sequence that students would typically encounter the concepts in school, i.e., her attention was drawn to a developmental, not necessarily conceptual, sequence. This observation epitomises the difference between the XLR8 curriculum and the typical school curriculum: XLR8 focuses upon conceptual structure and is less concerned with the traditions and incremental development of typical school curriculum. Using the graphical language – focussing on the cognitive transformations of knowledge – has perhaps assisted in focussing attention away from the typical developmentally based sequencing of instruction.

The genetic decompositions were then translated into a curriculum plan – a set of 16 modules to be delivered over two years. To construct this plan, a similar paper-based approach was used as this had proven to be a flexible technique. Using A3-sized paper, a matrix of 112 sheets was constructed (number, operations, algebra, measurement, geometry, statistics and probability sub-strands, for each of the 16 modules). During the process of constructing the genetic decompositions, several hundred concepts were identified. Of these, a sub-set of what were judged to be 'key' concepts from each sub-domain were identified, and new sticky-notes were created. Beginning first with number, operations and algebra sub-strands, these key concepts were then laid out in a chronological sequence, informed by the cognitively-based dependencies described in the corresponding genetic decompositions. As needed, concept sticky-notes were replicated, so that within each module, relevant concepts were identified, whether they were pre-requisite or those to be developed within that module. This process was then spread across the remaining sub-strands, and where necessary, concepts were re-sequenced so that inter-strand conceptual dependencies were satisfied. During that process of concept-sequencing, a parallel conversation regarding the overarching themes or contexts within which each modules concepts were to be housed and developed also helped to shape the structured sequence.

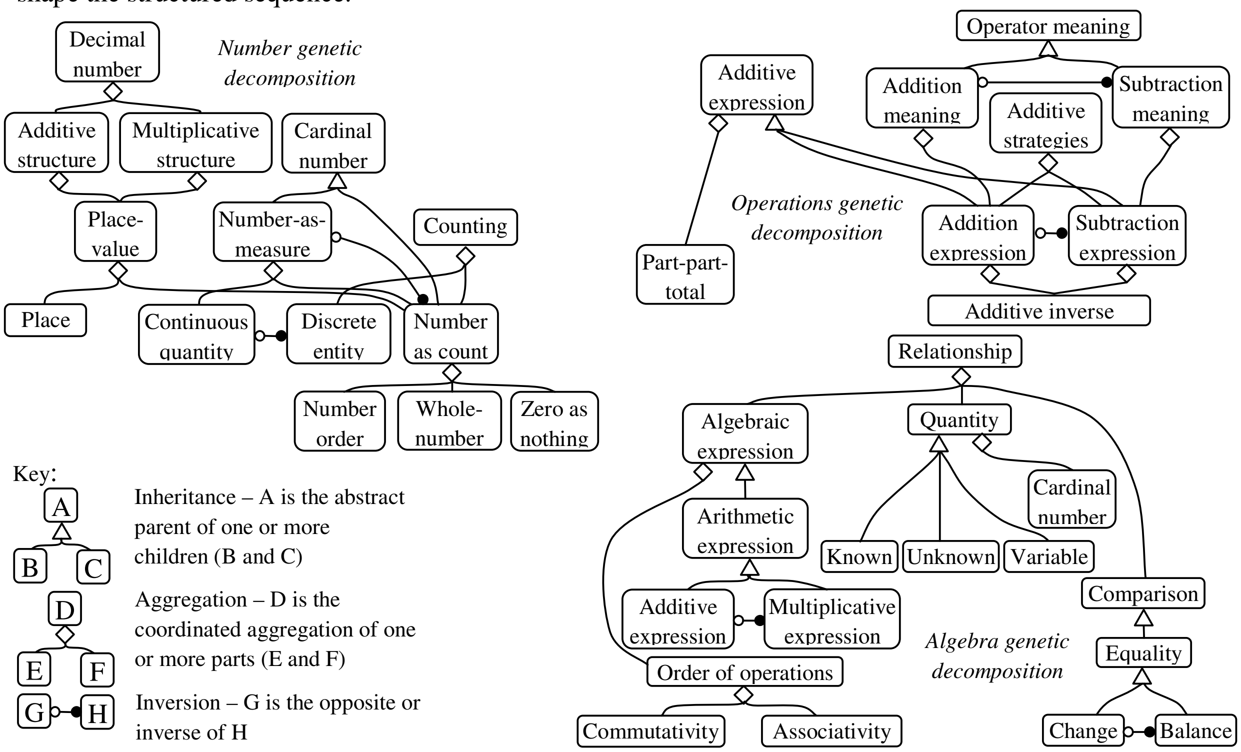

Figure 1 Excerpts of number, operation and algebra-specific genetic decompositions

Conclusion

This case study has reported upon the authors' use and reflective evaluation of genetic decomposition as a tool to document shared mathematical knowledge, and from that distil a conceptually sequenced curriculum. The set of concepts explored during and traditional curriculum documents usually only imply the conceptual structure. Instead, they tend to focus upon year-level specific performance expectations, providing teachers with a 'what to teach' rather than a 'why to teach' view of mathematics. To a certain degree, this has been the problem encountered in the XLR8 curriculum: the focus has been placed upon what to teach each lesson, rather than the reasons behind the lesson sequence. That is, the structure has remained largely tacit. By using the genetic decomposition technique, the authors have been forced to articulate to one another their understanding of the structure or inter-connections between concepts. This has then formed an overt base upon which to formulate the curriculum plan.

It is envisaged that the next step of the curriculum planning and development process will be to construct smaller, module-related genetic decompositions. That is, take the set of concepts identified as relevant to each module, identify the subset of associations that link them together (i.e., taken from the larger genetic decompositions already produced), and construct a genetic decomposition that described not only what (the concepts) and why (the interconnections) that should be developed during the teaching and learning of the module. This process could even be taken one step further, such that the module-level genetic decomposition could be broken into smaller pieces, each related to only a few lessons of the module. In this way, genetic decompositions could be used to scaffold the understanding of researchers, teachers and even students in relation to the organisation of mathematical ideas.

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